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LOCAL ISOMETRIC IMBEDDING OF RIEMANNIAN n-MANIFOLDS INTO EUCLIDEAN $(n+1)$-SPACE

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The problem of isometrically imbedding an n-dimensional Riemannian manifold M^n into Euclidean space E^{n+p} has received considerable attention. For example, it is now known that for each n, all infinitely differentiable M^n admit local isometric imbedding into $M^{n+(1/2)n(n+1)}$, and global isometric imbedding into $E^{n+p(n)}$, where $p(n)$ is a certain function whose optimal determination has been the object of recent study.

On the other hand, much less progress has been made in discovering necessary and sufficient conditions for a given M^n to be locally or globally isometrically imbeddable into E^{n+p} for various fixed values of $p < p(n)$. The known results are mostly limited to $p = 0$ and 1. The case $p = 0$ is of course classical—local isometric imbedding of M^n into E^n occurs when the curvature is zero, and global imbedding, when the global holonomy group is trivial. For $p = 1$, many conditions necessary for global imbedding are known, while sufficient conditions must await further local developments. The basic approach here is also classical. Namely, the fundamental theorem for hypersurfaces [2, p. 47] reduces the question of finding necessary and sufficient conditions for local isometric imbedding of M^n into E^{n+1} to the problem of solving the Gauss and Codazzi equations for a suitable second fundamental form tensor, in terms of the curvature tensor of M^n; therefore, the results obtained will necessarily be in the form of conditions on the curvature tensor.

The Gauss and Codazzi equations have been solved by T. Y. Thomas in his fundamental paper [4], and by N. A. Rozenson in her formidable work [3]. Each used different methods and obtained different types of conditions on the curvature tensor. Due to the quite complicated form of these results, however, the local $p = 1$ situation is far from being clear and warrants further work.

In the present paper, we use the method of bivectors and a theorem of W. L. Chow [1] to solve the Gauss (and Codazzi) equations in the case of a nonsingular curvature tensor, getting in this case, new necessary and sufficient conditions for local isometric imbedding of M^n into E^{n+1} (cf. Theorem 4 below).

We proceed with a precise statement of the problem, in our bivector setting. Let V be an n-dimensional real vector space with inner product. Let $\Lambda^2 V$ de-

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note the $\binom{n}{2}$-dimensional space of bivectors of V; it has an inner product induced from V by the definition $\langle x \wedge y, u \wedge v\rangle = \langle x, u\rangle\langle y, v\rangle - \langle x, v\rangle\langle y, u\rangle$. A linear map $L: V \to V$ defines a linear map $L \wedge L: \Lambda^2 V \to \Lambda^2 V$ by $(L \wedge L)(x \wedge y) = Lx \wedge Ly$; if L is symmetric, then so is $L \wedge L$. When V is taken to be the tangent space at a point of M^n, then the curvature tensor R at that point can be thought of as a symmetric linear map $R: \Lambda^2 V \to \Lambda^2 V$, via the equation $\langle R(x \wedge y), u \wedge v\rangle = \langle R(x, y)v, u\rangle$, where the R on the right side denotes the usual curvature operator on V, and the inner product in V is the Riemannian metric. Letting L denote the second fundamental form operator and denoting the covariant derivative by ∇, we can then express the Gauss and Codazzi equations as [2, pp. 30, 35]: $R = L \wedge L$ and ∇L is symmetric. Of these two equations, the Gauss equation $R = L \wedge L$ is the principal one, since under rather general conditions, Gauss implies Codazzi. Namely, (cf. [4, pp. 198, 205, and 191]) suppose M^n is a Riemannian manifold with C^3 metric, the rank of R (as a linear map on $\Lambda^2 V$) is ≥ 6, and the Gauss equation $R = L \wedge L$ holds at each point of M^n. Then the tensor field L (defined by the collection of L's at each point) is C^1 and satisfies the Codazzi equation. Thus the problem of locally isometrically imbedding into E^{n+1} a C^3 Riemannian manifold M^n with curvature of rank ≥ 6 is reduced to the following algebraic question: *Given a symmetric linear map $R: \Lambda^2 V \to \Lambda^2 V$, find necessary and sufficient conditions in order that there exist a symmetric linear map $L: V \to V$ satisfying $R = L \wedge L$.*

We now part company with the paths taken by Thomas and Rozenson, and exploit the bivector setting of the problem. Our first theorem uses a result of Chow [1] to establish the existence of a suitable L, modulo the right sign. Nonsingularity of the curvature R is essential to the argument.

Theorem 1. *Let R be nonsingular and symmetric, and let $n \geq 5$. Then there exists an L such that $R = \pm L \wedge L$ if and only if*

$$(1) \quad R(x_1 \wedge x_2) \wedge R(x_3 \wedge x_4) = -R(x_1 \wedge x_3) \wedge R(x_2 \wedge x_4) \qquad \textit{for all } x_i \in V\,.$$

Proof. If $R = \pm L \wedge L$, then (1) follows trivially. So it remains to show that (1) implies $R = \pm L \wedge L$. Let G_2 denote the subset of $\Lambda^2 V$ consisting of all nonzero decomposable bivectors, i.e., all α in $\Lambda^2 V$ having the form $\alpha = x \wedge y$, or equivalently, satisfying $\alpha \wedge \alpha = 0$ in $\Lambda^4 V$. Since $x \wedge y$ is nonzero if and only if x, y are independent, and since $u \wedge v$, $x \wedge y$ are proportional if and only if $\{u, v\} = \{x, y\}$, where $\{\cdots\}$ denotes the span of vectors in V, it follows that 2-dimensional subspaces of V correspond biuniquely with those 1-dimensional subspaces of $\Lambda^2 V$ which lie in the subset G_2.

Hence, if we pass to projective spaces $P(V)$ and $P(\Lambda^2 V)$, denoting the passage by square brackets, then $[G_2] \subset P(\Lambda^2 V)$ is precisely the Grassmann manifold of all projective lines in $P(V)$. We say $[\alpha], [\beta] \in [G_2]$ are adjacent if their corresponding projective lines in $P(V)$ intersect. Now Theorem I in [1, p. 38], with $r = 1$, can be stated in this way: *If* $f: [G_2] \to [G_2]$ *is a bijective mapping which preserves adjacence (both ways), and if* $\dim V \geq 5$, *then there exists a nonsingular linear map* $L: V \to V$ *such that* $f = [L \wedge L] | [G_2]$. (Remark: The dimension restriction serves to exclude correlations.)

Our nonsingular linear map $R: \Lambda^2 V \to \Lambda^2 V$ induces a bijection $[R]: P(\Lambda^2 V) \to P(\Lambda^2 V)$, and we want to apply Chow's result to $f = [R] | [G_2]$. In order to do this, we must verify that $[R]$ maps $[G_2]$ onto $[G_2]$ and preserves adjacence both ways.

To see what this means, we note the analytic meaning of adjacence. Namely, for $[\alpha], [\beta] \in [G_2]$, the corresponding projective lines in $P(V)$ are $[\{x, y\}]$ and $[\{u, v\}]$, where $\alpha = x \wedge y$, $\beta = u \wedge v$. These lines intersect if and only if $\dim \{x, y, u, v\} = 3$ if and only if $\alpha \wedge \beta = 0$ in $\Lambda^4 V$ if and only if α, β can be represented as $\alpha = a \wedge b$, $\beta = b \wedge c$. Therefore, if we can establish that

$$(2) \qquad \alpha \wedge \beta = 0 \quad \text{if and only if } R\alpha \wedge R\beta = 0\,, \quad \text{for all } \alpha, \beta \in \Lambda^2 V\,,$$

then it easily follows that $R(G_2) = G_2$ and that $[R] | [G_2]$ preserves adjacence both ways.

We shall now use our hypothesis (1) and the symmetry of R to establish condition (2). Consider the map $h: (V)^4 \to \Lambda^4 V$ defined by $h(x_1, x_2, x_3, x_4) = R(x_1 \wedge x_2) \wedge R(x_3 \wedge x_4)$; clearly h is multilinear, and (1) implies it is alternating. Hence h factors through a linear map $A: \Lambda^4 V \to \Lambda^4 V$, so that $A(x_1 \wedge x_2 \wedge x_3 \wedge x_4) = R(x_1 \wedge x_2) \wedge R(x_3 \wedge x_4)$. Consequently, $A(\alpha \wedge \beta) = R\alpha \wedge R\beta$ for all $\alpha, \beta \in \Lambda^2 V$. Since A is linear, $A(0) = 0$, which establishes (2) in one direction. The other part of (2) will follow from the nonsingularity of A, which we establish next, using the symmetry of R.

Namely, let w_r, $1 \leq r \leq \binom{n}{2}$, be a basis of $\Lambda^2 V$ which diagonalizes R, i.e., $Rw_r = \rho_r w_r$ for all r. Since R is nonsingular, all $\rho_r \neq 0$. Now the 4-vectors $w_r w_s$ span $\Lambda^4 V$, as can be seen from the expansions $e_i \wedge e_j \wedge e_k \wedge e_l = \sum_{r,s} x_{ij}^r x_{kl}^s w_r w_s$, where e_i, $1 \leq i \leq n$, is a basis of V, and $e_i \wedge e_j = \sum_r x_{ij}^r w_r$ is the basis change formula in $\Lambda^2 V$. Since a spanning set always contains a basis as a subset, it follows that some of the $w_r w_s$ form a basis of $\Lambda^4 V$. But $A(w_r w_s) = Rw_r Rw_s = \rho_r \rho_s w_r w_s$, so this basis diagonalizes A. Since all $\rho_r \rho_s \neq 0$, A must be nonsingular. This finishes the proof of (2); hence $[R] | [G_2]$ satisfies Chow's hypotheses.

We conclude, therefore, that there exists a nonsingular linear map $L: V \to V$ such that $[R] = [L \wedge L]$ on $[G_2]$. A standard technique of projective geometry can now be applied to show that $R = cL \wedge L$ for some constant $c \neq 0$. Namely, for each $x \wedge y \in G_2$, we have $[R(x \wedge y)] = [Lx \wedge Ly]$, whence $R(x \wedge y) = c_{x,y} Lx \wedge Ly$, with $c_{x,y} \neq 0$. Let us choose a basis $e_1, \cdots, e_n$ of V, and denote $c_{ij} = c_{e_i, e_j}$, so that $R(e_i \wedge e_j) = c_{ij} Le_i \wedge Le_j$. Note that $c_{ij} = c_{ji}$. Consider now the equation $R(e_i \wedge e_j) + R(e_i \wedge e_k) = R(e_i \wedge (e_j + e_k))$; the left side reduces to $c_{ij}(Le_i \wedge Le_j) + c_{ik}(Le_i \wedge Le_k)$, and the right to $c_{i,j+k}(Le_i \wedge Le_j) + c_{i,j+k}(Le_i \wedge Le_k)$. If $i \neq j$, $i \neq k$, $j \neq k$, then the bivectors $e_i \wedge e_j$ and $e_i \wedge e_k$ are independent. Also, $L \wedge L$ is nonsingular if and only if L is nonsingular. Therefore we can conclude that $c_{ij} = c_{i,j+k} = c_{ik}$ for all such i, j, k. Then the symmetry of c_{ij} implies that $c_{ij} = c_{kl}$ for all $i < j$ and $k < 1$; let us denote this common value of these c_{ij} by c. Thus we have $R(e_i \wedge e_j) = c(Le_i \wedge Le_j)$ for all $i < j$. Because $e_i \wedge e_j$ for $i < j$ is a basis of $\Lambda^2 V$, we get $R = cL \wedge L$, as was asserted above.

Since $c \neq 0$, we must have either $c > 0$ or $c < 0$. If we note that $cL \wedge L = \pm(\sqrt{|c|}\,L) \wedge (\sqrt{|c|}\,L)$, with the sign determined by that of c, we can rewrite $R = cL \wedge L$ as $R = \pm L \wedge L$ by redefining L as $\sqrt{|c|}\,L$. This finishes the proof of Theorem 1.

Our next task is to establish the symmetry of the map L obtained in Theorem 1, i.e., to prove that $\langle Lx, y\rangle = \langle x, Ly\rangle$ for all $x, y \in V$. For this purpose, we invoke the first Bianchi identity, which is satisfied at each point by the curvature R of each Riemannian manifold. In our notation, this identity appears as

$$\langle R(x \wedge y), z \wedge v\rangle + \langle R(z \wedge x), y \wedge v\rangle + \langle R(y \wedge z), x \wedge v\rangle = 0$$

for all $x, y, z, w \in V$.

Theorem 2. *Let R be nonsingular, let $n \geq 3$, and let R satisfy the first Bianchi identity. If $R = \pm L \wedge L$, then L must be symmetric.*

Proof. Let us substitute $R = \pm L \wedge L$ into the Bianchi identity and use the definition of inner product in $\Lambda^2 V$. We get, after collecting terms,

$$(3) \qquad \begin{aligned} &\langle Lx, v\rangle[\langle Lz, y\rangle - \langle Ly, z\rangle] + \langle Ly, v\rangle[\langle Lx, z\rangle - \langle Lz, x\rangle] \\ &\quad + \langle Lz, v\rangle[\langle Ly, x\rangle - \langle Lx, y\rangle] = 0 . \end{aligned}$$

Define the map $T\colon V \to E^3$ by $T(v) = (\langle Lx, v\rangle, \langle Ly, v\rangle, \langle Lz, v\rangle)$, where $x, y, z \in \in V$ are fixed. We clearly have

$$\ker T = \{Lx, Ly, Lz\}^{\perp} .$$

Since $R = \pm L \wedge L$ is nonsingular, so is L. Hence, if x, y, z are independent, then so are Lx, Ly, Lz. But then $\dim \{Lx, Ly, Lz\} = 3$, and consequently $\dim \ker T = n - 3$. Therefore $\operatorname{rank} T = 3$, i.e., T is onto; this means that, given any (a, b, c) in E^3, there is a v in V such that $a = \langle Lx, v\rangle$, $b = \langle Ly, v\rangle$, $c = \langle Lz, v\rangle$. If we let a, b, c be the three expressions inside the square brackets on the left side of (3), then (3) becomes $a^2 + b^2 + c^2 = 0$, whence $a = b = c = 0$. But $c = \langle Ly, x\rangle - \langle Lx, y\rangle$, so we get $\langle Lx, y\rangle = \langle x, Ly\rangle$, as was to be shown. On the other hand, if x, y are dependent, then $y = dx$, and $\langle Lx, y\rangle = \langle Lx, dx\rangle = \langle L(dx), x\rangle = \langle x, Ly\rangle$. This proves Theorem 2.

It remains to remove the minus sign from $R = \pm L \wedge L$. Let us observe first that the plus and the minus in $\pm L \wedge L$ denote two mutually exclusive classes of maps, namely, *if* $n \geq 3$ *and* L, M *are nonsingular, then* $L \wedge L \neq -M \wedge M$. (This follows from the proof on page 44 in [2] by inserting a minus sign; a contradiction will arise at the end: $1 + c^2 = 0$, c real).

We proceed to state a criterion to distinguish between the two classes of maps $+L \wedge L$ and $-L \wedge L$ on $\Lambda^2 V$. In order to do this, we must consider coordinate representations for R. If $e_1, \cdots, e_n$ is a basis of V, then $e_i \wedge e_j$, for $1 \leq i < j \leq n$, is a basis of $\Lambda^2 V$, and a linear map $R: \Lambda^2 V \to \Lambda^2 V$ has the coordinates R^{ij}_{kl}, $i < j$, $k < l$, with respect to this basis. These coordinates can be defined by $R^{ij}_{kl} = (e^i \wedge e^j)R(e_k \wedge e_l)$, where e^i denotes the dual basis of e_i. This formula in fact defines R^{ij}_{kl} for all values of i, j, k, l, but it is easy to see that the usual curvature identities hold:

$$R^{ij}_{kl} = -R^{ji}_{kl} = -R^{ij}_{lk} = R^{ji}_{lk} , \qquad R^{ii}_{kl} = R^{ij}_{kk} = 0 .$$

(If one does not want to use the dual basis, then one could, indeed, use these identities to define R^{ij}_{kl} for arbitrary i, j, k, l, from its values for $i < j$, $k < l$.) Define $\phi(R)$ and $\psi(R)$ by $\phi(R) = R^{ij}_{kl} R^{kp}_{iq} R^{lq}_{jp}$ and $\psi(R) = R^{ij}_{kl} R^{kl}_{pq} R^{pq}_{ij}$, where we sum over all repeated upper and lower indices. The functions $\phi(R)$ and $\psi(R)$ are scalar invariants of the tensor R^{ij}_{kl}, i.e., the coordinate expressions remain the same even if a basis change is performed. Hence any coodinates may be used to evaluate $\phi(R)$ and $\psi(R)$. (Remark: $\psi(R) = 8$ trace (R^3).)

Theorem 3. *Let* R *be nonsingular,* $R = \pm L \wedge L$, L *symmetric, and* $n \geq 3$.

(i) $R = L \wedge L$ *if and only if* $\phi(R) + \frac{1}{4}\psi(R) > 0$.

(ii) *In case* $n \equiv 3 \pmod 4$, $R = L \wedge L$ *if and only if* $\det R > 0$.

Proof. Since L is symmetric, there exists an orthonormal basis e_i in V which diagonalizes L: $Le_i = \lambda_i e_i$, for $i = 1, \cdots, n$. We know that $\lambda_i \neq 0$ for all i because L is nonsingular (since $\pm L \wedge L$ is). Then the basis $e_i \wedge e_j$, $i < j$, diagonalizes R:

$$R(e_i \wedge e_j) = \pm \lambda_i \lambda_j (e_i \wedge e_j) , \tag{4}$$

where $\pm$ means either $+$ always or $-$ always. Clearly the expression below is nonzero and is positive or negative according to the sign in (4), that is, the sign in $R = \pm L \wedge L$:

$$\sum_{i<j<k} (\pm \lambda_i \lambda_j)(\pm \lambda_i \lambda_k)(\pm \lambda_j \lambda_k) = \pm \sum_{i<j<k} (\lambda_i \lambda_j \lambda_k)^2 .$$

In the above orthonormal coordinate system, R has the matrix expression

$$R^{ij}_{kl} = \pm \lambda_i \lambda_j \delta^{ij}_{kl} , \qquad i < j , \ k < l . \tag{5}$$

If the values R^{ij}_{kl} are defined for all i, j, k, l according to the customary scheme mentioned in the discussion above, then we can evaluate the expression for

$\phi(R)$ by summing it out over i, j, k, l, p, q. A long but straightforward calculation shows that

$$\phi(R) = \pm 8 \sum_{i<j<k} (\lambda_i\lambda_j\lambda_k)^2 - 2 \sum_{i<j} (\pm\lambda_i\lambda_j)^3 ,$$

where $+$ or $-$ is taken as $R = L \wedge L$ or $R = -L \wedge L$, respectively. A much shorter calculation yields $\psi(R) = 8 \sum_{i<j} (\pm\lambda_i\lambda_j)^3$. Hence $\phi(R) + \frac{1}{4}\psi(R) = \pm 8 \sum_{i<j<k} (\lambda_i\lambda_j\lambda_k)^2$. Thus we see that the sign of $\phi(R) + \frac{1}{4}\psi(R)$ does indeed distinguish between $R = L \wedge L$ and $R = -L \wedge L$.

To prove part (ii), we can use the same special coordinate system and the matrix expression (5) to calculate that

$$\begin{aligned} \det R &= (\pm\lambda_1\lambda_2)(\pm\lambda_1\lambda_3)\cdots(\pm\lambda_{n-1}\lambda_n) \\ &= (\pm 1)^{\binom{n}{2}}(\lambda_1\lambda_2\cdots\lambda_n)^{n-1} = (\pm 1)^{\binom{n}{2}}(\det L)^{n-1} . \end{aligned}$$

If $n = 4s + 3$, then $\binom{n}{2} = \frac{1}{2}n(n-1)$ is odd and $n - 1$ is even. Hence, noting that $\det L \neq 0$, we see $\det R = \pm$(positive number). Since the $+$ or $-$ is taken as $R = L \wedge L$ or $R = -L \wedge L$, respectively, the criterion (ii) is established, and Theorem 3 is proved.

Our main theorem about local isometric imbedding now follows directly from the results proved above. We assume the manifold has a metric of class C^3, at least.

Theorem 4. *Let M^n, with $n \geq 5$, be a Riemannian manifold with non-singular curvature tensor R. Then M^n admits local isometric imbedding into E^{n+1} if and only if*

(i) $R(x_1 \wedge x_2) \wedge R(x_3 \wedge x_4) + R(x_1 \wedge x_3) \wedge R(x_2 \wedge x_4) = 0$, *for all* $x_i \in V$, *and*

(ii) $R^{ij}_{kl}R^{kp}_{iq}R^{lq}_{jp} + \frac{1}{4}R^{ij}_{kl}R^{kl}_{pq}R^{pq}_{ij} > 0$.

Moreover, if $n \equiv 3 \pmod 4$, *then* (ii) *can be replaced by* $\det R > 0$.

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